Trigonometry
Solving Triangles: The Law of Cosines

The Law of Cosines is useful when you are given information about all three sides of the triangle or information about two sides and the included angle. Basically, to use the Law of Cosines you need information about all three side or two sides and an included angle.

Law of Cosines:

Let ΔABC be any triangle with a, b, and c representing the measures of sides that are opposite angles with measures A, B, and C respectively. Then the following are true:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

But there is one thing that you need to remember about triangles: the sum of any two sides must be greater than the third side. So if you are given all three sides of a triangle, add the two smallest sides together first to make sure that those sides can actually form a triangle. Sometimes the problem tries to trick you.

Example 1: Solve ΔABC if $a = 67$, $b = 54.1$ and $c = 94.9$. We are given the three side lengths; we need to find the measures of all three angles. Check to see if the measures work for a triangle:

$$67 + 54.1 > 94.9$$
$$67 + 94.9 > 54.1$$
$$54.1 + 94.9 > 67$$

We should start by finding the largest angle first, let's find angle C:

$$94.9^2 = 67^2 + 54.1^2 - 2\cdot 67\cdot 54.1\cdot\cos C$$
$$9006.01 = 7415.81 - 7249.4\cos C$$
$$1590.2 = -7249.4\cos C$$
$$\frac{1590.2}{-7249.4} = \cos C$$
$$\cos^{-1}\left(\frac{1590.2}{-7249.4}\right) = C$$
$$102.7^\circ = C$$

Be sure that you do NOT subtract 7415.81 and 7249.4! Remember the order of operations!

Example2: Solve ΔABC if $a = 170.5$, $b = 52$, and $c = 72$.

$$52 + 72 < 170.5$$

No triangle can be formed.

The law of cosines can also be used when you are given two sides of the triangle and the angle between them.

Example3: Solve ΔABC if $b = 102$, $c = 64.5$, and $A = 111°$

We must find side a first:

$$\begin{aligned} a^2 &= 102^2 + 64.5^2 - 2 \cdot 102 \cdot 64.5 \cdot \cos 111 \\ a^2 &= 14564.25 - (-4715.40548) \\ a^2 &= 19279.65548 \\ a &= 138.9 \end{aligned}$$

Now use the law of cosines to find another angle, we will find angle B:

$$\begin{aligned} 102^2 &= 138.9^2 + 64.5^2 - 2 \cdot 138.9 \cdot 64.5 \cdot \cos B \\ 10404 &= 23439.905 - 17911.805 \cos B \\ -13035.905 &= -17911.805 \cos B \\ \frac{-13035.905}{-17911.805} &= \cos B \\ 43.3° &= B \end{aligned}$$

And the last angle is:

$$C = 180 - 111 - 43.3 = 25.7°$$

The largest side (a) was across from the largest angle (A) and the smallest side (c) was across from the smallest angle (C), so we can be confident in our solution.